Journal of Pure and Applied Algebra 43 (1986) 271-279
North-Holland

SEPARABLY REAL CLOSED LOCAL RINGS

André JOYAL
Dept. of Mathematics, University of Sydney, Sydney, Australia

Gonzalo E. REYES*
Dept. of Mathematics, Université du Quebec à Montreal, Canada

Communicated by M. Barr
Received 7 December 1984

It is well known that several types of 'variable reals' arising in topos theory (Dedekind reals, Cauchy reals, smooth reals, etc.) are not real closed, in the sense that a polynomial (with 'variable real' coefficients) may change sign in an interval without having a zero in that interval.

This phenomenon stems from the fact that roots of polynomials are not, in general, continuous functions of the coefficients, as the example of the cubic x^3+px+q shows. Indeed, there is no continuous function $x(p,q)$ such that

$$x(p,q)^3+px(p,q)+q=0$$

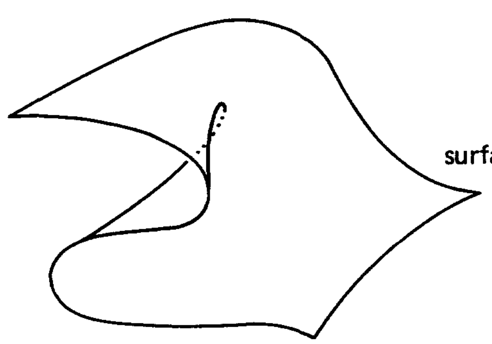
in a neighbourhood of $p=0$, $q=0$.

This is most easily visualized by looking at the catastrophe map χ ('the cusp')

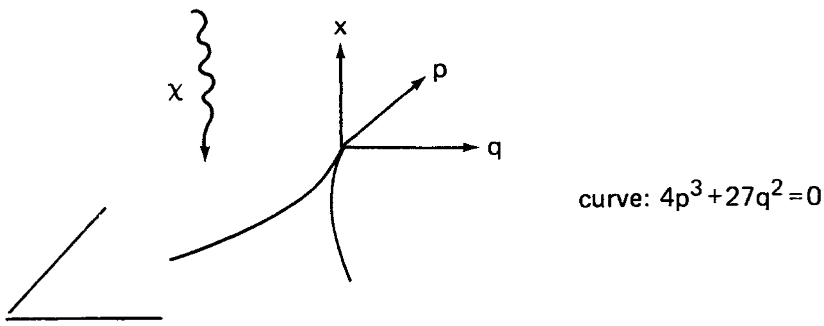
$$\chi:\{(p,q,x)\in\mathbb{R}^3\,|\,x^3+px+q=0\}\to\mathbb{R}^2$$

defined by

$$\chi(p,q,x)=(p,q).$$



* Research partially supported by the Natural Sciences and Engineering Research Council of Canada, and the Ministère de l'Education, Gouvernement du Québec.

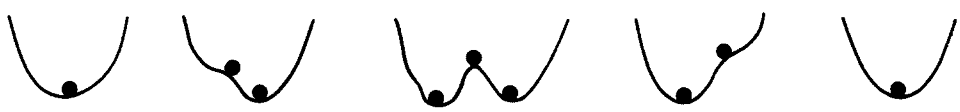


It is obvious that χ has no continuous sections in any neighbourhood of $p=0$, $q=0$ (just go around a circle with centre $(0,0)$ in the (p,q)-plane).

This phenomenon has the following physical interpretation: the positions of equilibrium of a dynamical system given by the potential

$$V=\frac{x^4}{4}+\frac{px^2}{2}+qx$$

do not depend continuously on the parameters p,q. In pictures:



For more information on this subject, the reader may consult Poston–Stewart [8].

The question then arises as to the appropriate topos-theoretic notion of 'real closed field'. Kock [5] has proposed the notion of 'separably real closed local ring', meaning a commutative ring with unit which is local, Henselian and has a real closed residue field. We recall (see, e.g., Raynaud [9]) that a ring A is *local* if it has exactly one proper maximal ideal m_A. A is *Henselian* if any simple root in its residue field $k_A=A/m_A$ of a polynomial $p\in A[t]$ can be lifted to a (necessarily unique) root of p in A.

To show the appropriateness of his notion, Kock [5] proves that

(i) It generalizes the notion of a real closed field, which is just a separably real closed field.

(ii) Various sheaves of continuous, C^∞, analytic, ... real functions in appropriate spatial toposes are examples of this notion.

(iii) It is coherent (see, e.g., Johnstone [3] or Makkai–Reyes [7] for this notion).

(iv) It is ε-stable (or infinitesimally stable) in the sense of Kock [5].

Kock [5] also conjectured that the object of Dedekind reals, in an arbitrary elementary topos with a natural number object, is a separably real closed local ring (object). For the particular case of Grothendieck toposes, this conjecture was verified by Johnstone [4].

In this paper we prove Kock's conjecture, as well as related results about Cauchy reals and smooth reals.

Our main tool is a strengthening of Tarski's theorem on elimination of quantifiers in real closed fields, due to Coste and Coste-Roy, Delzell, Bocknak, and Efroymson (see, e.g., Coste & Coste-Roy [2]). This result provides a new coherent axiomatization of the notion of the title which is the key for the whole proof.

Throughout the paper, we shall use the set-theoretical language as described, e.g., in Boileau-Joyal [1].

1. The coherent axiomatization

We say that a local ring A is *ordered* if it has an order relation, $<$, which is compatible with the ring operations and induces a linear order in its residue field $k_A = A/m_A$. More explicitly, $<$ is assumed to satisfy the following axioms:

$$1>0,$$

$$x>0 \wedge y>0 \rightarrow x+y>0 \wedge x\cdot y>0,$$

$$x \text{ invertible} \leftrightarrow x>0 \vee -x>0.$$

We define $x>y \leftrightarrow x-y>0$.

Our axiomatization will be formulated in the language L of the theory of ordered rings with $+, -, \cdot, 0, 1, >$ as non-logical symbols.

If A is any ordered local ring and ϕ any formula of L whose free variables are among $x=(x_1, \dots, x_n)$, we let

$$A_x(\phi) = \{a \in A^n \mid A \models \phi[a]\}$$

be the 'extension of ϕ in A'.

The strengthening of the theorem in Tarski mentioned in the introduction is the following.

Theorem. *Let ϕ be a formula of L whose free variables are among $x=(x_1, \dots, x_n)$. Assume that $\mathbb{R}_x(\phi) \subset \mathbb{R}^n$ is open (in the usual topology). Then there is a formula ϕ_0 having the same free variables of ϕ and of the form*

$$\bigvee_i \bigwedge_j P_{ij}(x) > 0$$

where the P_{ij} are terms (i.e. polynomials) and such that

$$K \models \forall x\,(\phi \leftrightarrow \phi_0), \quad \textit{for any real closed field } K.$$

The *crucial* property of such a formula ϕ_0 is this: it is preserved and reflected by local homomorphisms (i.e. homomorphisms which reflect invertible elements) between ordered local rings.

To state our main result, we shall identify a monic polynomial $p(t) = t^n + p_1 t^{n-1} + \cdots + p_n$ with the sequence $p = (p_1, \dots, p_n)$ of its coefficients.

Consider the following formula of L:

$$\delta(p) \equiv p(0)p(1) < 0 \wedge p'(0) > 0 \wedge p'(1) > 0 \wedge \forall x\,(0 < x < 1 \to p'(x) > 0).$$

Using the previous theorem and the fact that $\mathbb{R}_p(\delta) \subset \mathbb{R}^n$ is open (in the usual topology), there is another formula δ_0 of L of the form $\bigvee_i \bigwedge_j P_{ij}(p) > 0$ such that

$$K \models \forall p\,(\delta \leftrightarrow \delta_0), \quad \text{for any real closed field } K.$$

Proposition. *If A is any ordered local ring, then*

$$A \models \forall p\,(\delta_0 \to \delta).$$

Proof. Assume that $A \models \delta_0[p]$. From the theorem and the crucial property of δ_0 we conclude that $\bar{k}_A \models \delta[(i \circ \tau_A)(p)]$ where $A \xrightarrow{\tau_A} k_A$ is the canonical map and $k_A \overset{i}{\rightarrowtail} \bar{k}_A$ is the inclusion of k_A in its real closure $\bar{k}_A$. A fortiori, $k_A \models \delta[\tau_A(p)]$ and this implies that $A \models \delta[p]$. $\square$

Theorem. *The following is a (coherent) axiomatization of the notion of a separably real closed local ring:*

(i) *Commutative ring with unit axioms.*

(ii) *Localness:*

$$\begin{cases} \neg(0=1), \\ x \text{ invertible} \vee (1-x) \text{ invertible.} \end{cases}$$

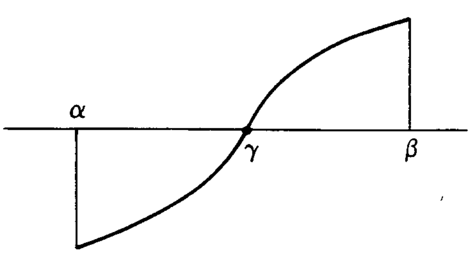
(iii) *Order:*

$$\begin{cases} 1 > 0, \\ x > 0 \wedge y > 0 \to x + y > 0 \wedge x \cdot y > 0, \\ x \text{ invertible} \leftrightarrow x > 0 \vee -x > 0. \end{cases}$$

(iv) $\delta_0(p) \to \exists x\,(0 < x < 1 \wedge p(x) = 0 \wedge p'(x) > 0)$.

Notice that, since $p = (p_1, \ldots, p_n)$ is a sequence of n elements, (iv) is actually an infinite set of axioms, one for each $n = 1, 2, \ldots$.

Proof. Assume (i)–(iv). Let $p \in k_A[t]$ be either $p = t^2 - \delta$, with $\delta > 0$ or a monic polynomial of odd degree such that $(p, p') = 1$. Going over to the real closure $k_A \overset{i}{\rightarrowtail} \bar{k}_A$, we find a simple root γ of p in $\bar{k}_A$. In pictures:



for some interval (α, β) whose end points may be assumed to be in k_A, by continuity of p, such that $p' > 0$ throughout that interval (this is always possible, by changing p to $-p$, if necessary). Using the transformation $u \mapsto \alpha + (\beta - \alpha)u$, we may assume that $\alpha = 0$, $\beta = 1$, $\gamma \in (0, 1)$, i.e.,

$$\bar{k}_A \models \delta_0[i(p)].$$

By the crucial property of δ_0 (reflection under local homomorphisms between ordered local rings)

$$A \models \delta_0[\tilde{p}],$$

where $\tilde{p} \in A[t]$ is any lifting of p to A.

From (iv), $\tilde{p}$ has a simple root in A whose image in k_A is a (simple) root of $p \in k_A[t]$. This shows that k_A is real closed.

To show that A is Henselian, let $\alpha \in k_A$ be a simple root of $p \in A[t]$. Since k_A is real closed (as just proved), $k_A \models \delta_0[\tau_A(p)]$ (using a transformation of the form $u \mapsto \alpha + (\beta - \alpha)u$ again). Hence $A \models \delta_0[p]$ and so there is a simple root $a \in A$ of p. If $\tau_A(a) = \alpha$, there is nothing else to prove; if not, write $p = (t - a)q$ in $A[t]$ and use induction on the degree of p.

The proof in the other direction is obvious. □

2. Applications to variable reals

Corollary 1 (Kock's conjecture). *The object $\mathbb{R}_D$ of Dedekind reals in any elementary topos with an object of natural numbers is a separably real closed local ring.*

Proof. It is well known that $\mathbb{R}_D$ satisfies axioms (i)–(iii) (see, e.g. Johnstone [3]). Assume that $\mathbb{R}_D \models \delta^0[p]$. Since $\delta_0 \to \delta$ is (equivalent to) a coherent sequent and is true for all local ordered rings (whose theory is coherent), then it is true in $\mathbb{R}_D$ by the Metatheorem of Makkai–Reyes [7]. Therefore $\mathbb{R}_D \models \delta[p]$ and the result follows from the following.

Lemma. *Let $f \in \mathbb{R}_D^{\mathbb{R}_D}$ be a locally uniformly continuous function such that*

(i) $x < y \to f(x) < f(y)$,

$$\left.\begin{array}{l}\text{(ii) } f(\infty) = \infty \\ \text{(iii) } f(-\infty) = -\infty\end{array}\right\} \text{ in the obvious sense.}$$

Then f is a homeomorphism.

Proof. Just check that, for each $r \in \mathbb{R}_D$, the pair

$$\{a \in \mathbb{Q} \mid f(a) < r\}, \qquad \{a \in \mathbb{Q} \mid f(a) > r\}$$

constitutes a Dedekind cut.

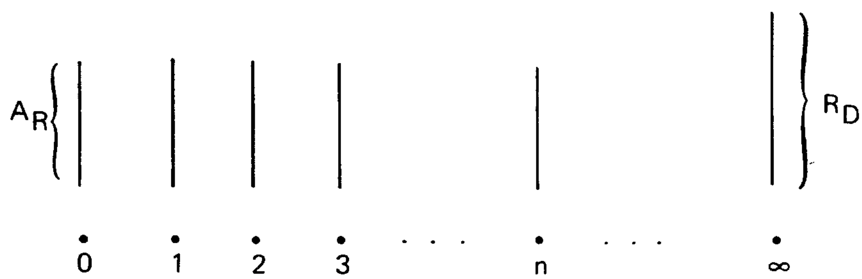
Corollary 2. *The (object of) Cauchy reals, $\mathbb{R}_C$, in any elementary topos $\mathscr{E}$ with natural number object is a separably real closed local ring.*

Proof. Let $\mathrm{Sh}(\mathbb{N} \cup \{\infty\})$ be the $\mathscr{E}$-topos of sheaves over the Alexandroff compactification of $\mathbb{N} \in \mathscr{E}$. The (object of) Dedekind reals (resp. the algebraic reals) will be denoted, in both toposes, by $\mathbb{R}_D$ (resp. $\mathbb{A}_{\mathbb{R}}$).

Let $R \in \mathrm{Sh}(\mathbb{N} \cup \{\infty\})$ be defined by $R = \{x \in \mathbb{R}_D \mid [\![x \in \mathbb{A}_{\mathbb{R}}]\!] \supseteq \mathbb{N}\}$. More precisely,

$$R(V) = \{x \in C^0(V, \mathbb{R}_D) \mid \forall x \in \mathbb{N} \cap V\ a(x) \in \mathbb{A}_{\mathbb{R}}\} \quad \text{for all } V \in \mathcal{O}(\mathbb{N} \cup \{\infty\}).$$

As an étale space, R has the following representation:



It is immediate that $R_\infty \in \mathscr{E}$, the germ at ∞, is isomorphic to the ring of Cauchy sequences of $\mathbb{A}_{\mathbb{R}}$ modulo de Frechet filter of cofinite subsets of $\mathbb{N}$. Since every algebraic real has a decimal expansion, a diagonal argument shows that the quotient of R by the Cauchy sequences converging to 0 may be identified with the Cauchy reals:

$$R_\infty \twoheadrightarrow \mathbb{R}_C \in \mathscr{E}.$$

Furthermore, since separably real closed local rings are closed under non-trivial quotients, it is enough to prove that R_∞ is such a ring. On the other hand, since this notion is coherent and the operations of taking germs preserves coherent logic, we only need to prove that $R \in \mathrm{Sh}(\mathbb{N} \cup \{\infty\})$ is separably real closed. This is a consequence of the following.

Lemma. *Let $R_1 \rightarrowtail R_2$ be a local monomorphism between separably real closed local rings. Then $R = \{x \in R_2 \mid [\![x \in R_1]\!] \supseteq U\}$ is again such a ring, for all $U \rightarrowtail 1$.*

Proof. Only axiom (iv) needs a proof.

Let $p \in R[t]$ be a monic polynomial of degree n such that $R \models \delta_0[p]$. Since R_2 is separably real closed, the unique root in $(0,1) \subset R_2$ of any monic polynomial satisfying δ_0 defines a morphism

$$\xi : \{p \in R_2^n \mid R_2 \models \delta_0[p]\} \to R_2$$

whose restriction to R_1^n lies in R_1 (given that R_1 is separably real closed).

But $[\![p \in R_1^n]\!] \supset U$ and so $[\![\xi(p) \in R_1]\!] \supseteq [\![p \in R_1^n]\!] \supseteq U$, i.e., the unique root of p in $(0,1)$ lies in R. $\square$

To formulate our next result, let $C^\infty \in \mathscr{E}$ be the (internal) theory $C^\infty = \{\mathbb{R}^n : n \in \mathbb{N}\}$ whose n-ary operations are $C^\infty(\mathbb{R}^n, \mathbb{R}) \subset \mathbb{R}^{\mathbb{R}^n}$, the (internal) smooth functions from $\mathbb{R}^n$ into $\mathbb{R}$. (We are using $\mathbb{R}$ to denote the object of Dedekind reals in $\mathscr{E}$).

A *C^∞-ring* in $\mathscr{E}$ is an (internal) functor $A \in \mathscr{E}^{C^\infty}$ which preserves products. A is *local* if and only if $A(\mathbb{R})$ is a local ring (object) in $\mathscr{E}$.

Corollary 3. *Any local C^∞-ring in $\mathscr{E}$ is separably real closed.*

Proof. There are two sources of difficulties in this proof. First, the theory of separably real closed local rings was formulated by using $<$ as a non-logical relation symbol. We need to relate $>$ with smooth functions, which are all what C^∞-rings know about. This is easily done, by constructing (as usual) a 'characteristic' function of the open set $\{x \in \mathbb{R} \mid x > 0\}$, i.e., a function $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ such that

$$x > 0 \quad \text{iff} \quad \chi(x) \text{ invertible.}$$

We now reformulate the axioms for a separably real closed local rings in terms of χ:

(i), (ii): As before.

(iii) $\chi(1)$ invertible,

$\chi(x)$ invertible $\wedge\, \chi(y)$ invertible $\to \chi(x+y)$ invertible $\wedge\, \chi(x \cdot y)$ invertible,

x invertible $\to \chi(x) + \chi(-x)$ invertible,

$\chi(x)$ invertible $\to x$ invertible.

(iv) Replace "$x > 0$" by "$\chi(x)$ invertible" throughout.

Now comes the second difficulty: our axioms are not equations and hence do not hold, a priori, in a C^∞-ring. Our solution (Lemma 3) is to show that axioms of this type are consequences of equations true in $\mathbb{R}$.

We need some auxiliary results.

Lemma 1 (Existence of bump functions). *For all $\varepsilon \in \mathbb{R}$, $\varepsilon > 0$ there is $r_\varepsilon \in C^\infty(\mathbb{R}, \mathbb{R})$ such that*

$$r_\varepsilon(x) \begin{cases} = 0 & \text{if } |x| > \varepsilon, \\ > 0 & \text{if } |x| < \varepsilon, \\ = 1 & \text{if } x = 0. \end{cases}$$

In particular, $r_\varepsilon(x)$ is invertible if and only if $|x| < \varepsilon$.

Proof. The usual proof which starts from the function

$$f(x) = \begin{cases} e^{-1/x^2} & \text{if } x > 0, \\ 0 & \text{if } x \le 0 \end{cases}$$

is constructive and valid in $\mathscr{E}$. Furthermore, such an f is clearly defined on $\mathbb{Q}$ and is uniformly continuous. Hence, it has a unique extension to $\mathbb{R}$. $\square$

Lemma 2. *Let $\phi \in C^\infty(\mathbb{R}^n)$ and $U = \{x \in \mathbb{R}^n \mid \phi(x)$ invertible$\}$. Then*

$$C^\infty(\mathbb{R}^{n+1})/(y\phi(x)-1) \simeq C^\infty(U).$$

Proof. Let $\varrho: C^\infty(\mathbb{R}^n) \to C^\infty(U)$ be the 'restriction' map defined by $\varrho f(x,y) = f(x, 1/\phi(x))$. We claim that ϱ is surjective, i.e., any $h \in C^\infty(U)$ may be 'extended' to some $f \in C^\infty(\mathbb{R}^{n+1})$. The argument below was suggested by Ngo van Quê.

Let $h \in C^\infty(U)$. Define, for $\varepsilon > 0$,

$$f(x,y) = \begin{cases} r_\varepsilon\left(y - \dfrac{1}{\phi(x)}\right)h(x) & \text{if } \phi(x) \text{ invertible,} \\ 0 & \text{if } |y\phi(x)-1| > \varepsilon|\phi(x)|. \end{cases}$$

Using the fact that $\mathbb{R}$ is an ordered local ring, one easily checks that $f \in C^\infty(\mathbb{R}^{n+1})$. Indeed, either $|\phi(x)| < 1/(\varepsilon + |y|)$ and hence $|y\phi(x)-1| > \varepsilon|\phi(x)|$ or $|\phi(x)| > 1/2(\varepsilon + |y|)$, which implies that $\phi(x)$ invertible.

Clearly, $h(x) = f(x, 1/\phi(x))$ for all $x \in U$.

Assume now that $f \in \operatorname{Ker}(\varrho)$. Using Hadamard's lemma for $f \in C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R})$:

$$f(x,t) - f(x,s) = (t-s)\int_0^1 \frac{\partial f}{\partial t}(x, s+(t-s)u)\,\mathrm{d}u$$

we conclude the existence of some $f_1 \in C^\infty(\mathbb{R}^{n+2})$ such that

$$f(x,y) - f\left(x, \frac{1}{\phi(x)}\right) = \left(y - \frac{1}{\phi(x)}\right) f_1\left(x, y, \frac{1}{\phi(x)}\right).$$

Define

$$v(x,y) = \begin{cases} \dfrac{f(x,y)}{y\phi(x)-1} & \text{if } y\phi(x)-1 \text{ invertible,} \\[2ex] \dfrac{f_1(x,y,1/\phi(x))}{\phi(x)} & \text{if } \phi(x) \text{ invertible.} \end{cases}$$

Once again, it is easily checked that $v \in C^\infty(\mathbb{R}^{n+1})$. Therefore,

$$f(x,y) = v(x,y)(y\phi(x)-1) \in (y\phi(x)-1). \quad \square$$

Lemma 3. *Let $\phi(x)$, $\theta(x,z)$, $\psi(x,t)$ be smooth functions. Assume the exisence of some $h \in C^\infty(U)$ such that $\theta(x, h(x)) = 0$ and $\psi(x, h(x))$ is invertible, for all $x \in U$, where $U = \{x \in \mathbb{R}^n \mid \phi(x)$ invertible$\}$. Then any C^∞-ring satisfies the sentence*

$$\forall x\,(\phi(x) \textit{ invertible} \to \exists z\,(\theta(x,z) = 0 \wedge \psi(x,z) \textit{ invertible})).$$

Proof. By Lemma 2, we can 'extend' $h(x)$ to $f(x,y)$ in such a way that

$$\theta(x, f(x,y)) = 0 \quad \text{modulo } (y\phi(x)-1)$$

and

$$\psi(x, f(x,y)) \text{ invertible} \quad \text{modulo } (y\phi(x)-1).$$

In other words, there are smooth functions v, v_1, ψ_1 such that the following equations are valid in $\mathbb{R}$:

$$\begin{cases}\theta(x, f(x, y)) = v(x, y)(y\phi(x) - 1),\\ \psi(x, f(x, y))\psi_1(x, y) - 1 = v_1(x, y)(y\phi(x) - 1).\end{cases}$$

Any C^∞-ring is a model of these equations. □

To finish the proof of Corollary 3, we notice that all axioms for separably real closed local rings, as reformulated in terms of χ, are of the form stated in Lemma 3. The result thus follows from Corollary 1. □

References

[1] A. Boileau and A. Joyal, La logique des topos, J. Symbolic Logic 46 (1) (1981).

[2] M.-F. Coste-Roy and M. Coste, Topologies for real algebraic geometry, in: Topos theoretic Methods in Geometry, Various Publications Series 30 (Aarhus, 1979).

[3] P. Johnstone, Topos Theory, London Math. Soc. Monographs 10 (Academic Press, New York, 1977).

[4] P. Johnstone, Extracts from two letters to A. Kock, in: Topos theoretic Methods in Geometry, Various Publication Series 30 (Aarhus, 1979).

[5] A. Kock, Formally real local rings and infinitesimal stability, in: Topos theoretic Methods in Geometry, Various Publications Series 30 (Aarhus, 1979).

[6] A. Kock, Synthetic differential geometry, London Math. Soc. Lecture Notes Series 51 (Cambridge Univ. Press, Cambridge, 1981).

[7] M. Makkai and G.E. Reyes, First-order categorical logic, Lecture Notes in Math. 611 (Springer, Berlin, 1977).

[8] T. Poston and I. Stewart, Catastrophe Theory and its Applications (Pitman, London, 1978).

[9] M. Raynaud, Anneaux locaux Henseliens, Lecture Notes in Math. 169 (Springer, Berlin, 1970).

[10] G.E. Reyes (editor), Géométrie différentielle synthétique, Fasc. 1, Schémas lisses, Rapports de Recherches DMS (Montréal, 1980).